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An Unsolved Problem of Fenchel

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Abstract. The Fenchel problem of level sets is solved under the conditions that the boundaries of the nested family of convex sets in R^{n+1} are given by C^3 n-dimensional differentiable manifolds and the convex sets determine an open or closed convex set in R^{n+1}.

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1. Introduction

It is well-known that a convex function has convex less-equal level sets. That the converse is not true was realized by de Finetti (1949). The problem of level sets, discussed first by Fenchel in 1953, is as follows: Under what conditions is the family of level sets of a convex function a nested family of closed convex sets? Fenchel (1953, 1956) gave necessary and sufficient conditions for the existence of a convex function with the prescribed level sets and the existence of a smooth convex function under the assumption that the given subsets are the level sets of a twice differentiable function. In the first case, seven conditions were deduced, and while the first six are simple and intuitive, the seventh is rather complicated. This fact and the additional assumption in the smooth case, according to which the given subsets are the level sets of a twice differentiable function, seem to be the motivation that Roberts and Varberg (1973, p.271) drew up anew the following problem of level sets: "What "nice" conditions on a nested family of convex sets will ensure that it is the family of level sets of a convex function?" In the sequel, the notions of convexifiability and concavifiability are used as synonyms, because if a function f is convex, then $-f$ is concave.

In theory of economics, Debreu (1954) proved his famous theorem on the representation of a continuous and complete preference ordering by a utility function. It is obvious that the utility function, whose existence is given by the Debreu theorem, is quasiconcave if the preference ordering is convex. Crouzeix (1977) and Kannai (1977, 1981) studied the problem of concavifiability of convex preference orderings, i.e., the problem of the existence of a concave function having the same level sets as a given continuous quasiconcave one and they improved the Fenchel

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results. This problem can be important in several economic and bargaining situations. The conditions provided for the cases of continuous, differentiable and twice differentiable quasiconcave functions are intimately related to constructions of special (least concave) utility representations. (By Debreu (1976), a utility function is said to be least concave on a convex set if every concave utility function defined on the same set can be represented by a concave transformation of the given utility function.) Crouzeix (1977) and Kannai (1981) introduced auxiliary functions, observing that the concavifiability of a quasiconcave function is essentially a one-dimensional phenomenon, and that if the convex preference ordering is concavifiable, then a suitably constructed auxiliary quasiconcave utility function has to possess finite and non-vanishing one-sided directional derivatives.

An unusual feature of concavifiability theory, as presented in Kannai (1977), was the use of Perron's integral in expressing concavifiability in terms of second-order (one-point) conditions involving a twice differentiable quasiconcave utility function. It turns out that in case a function like this exists at all, the auxiliary function is also twice differentiable, and the associated function, whose Perron integrability is equivalent to concavifiability, has a constant sign, hence Perron integrability is equivalent (in the term of auxiliary functions) to Lebesque integrability (Crouzeix, 1977 and Kannai, 1981).

Rapcsák (1991) gave an explicit formulation of the gradient of the class of the smooth pseudolinear functions (both pseudoconvex and pseudoconcave). This result means an extension of the Cauchy functional equation and the solution of the Fenchel problem in the case of a nested family of convex sets whose boundaries are of hyperplanes which define an open convex set, and if this family of convex sets corresponds to the less-equal level sets of smooth pseudolinear functions. We have to point out that the pseudolinear functions are contained in a special class of functions which is more general than the class of linear functions (both convex and concave). The following theorem was proved for characterizing the gradient of the smooth pseudolinear functions:

THEOREM 1.1 (Rapcsák, 1991a). *Let a three times continuously differentiable function f be defined on an open convex set $A \subseteq R^n$ and assume that $\nabla f(\mathbf{x}) \neq 0$, $\mathbf{x} \in A$. Then, f is pseudolinear on A iff there exist twice continuously differentiable functions $l(\mathbf{x})$, $\eta_i(f(\mathbf{x}))$, $i = 1, \ldots, n$, $\mathbf{x} \in A$, such that the following conditions are satisfied:*

$$\frac{\partial f(\mathbf{x})}{\partial x_i} = l(\mathbf{x})\eta_i(f(\mathbf{x})), \quad i = 1, \ldots, n, \quad \mathbf{x} \in A. \tag{1.1}$$

In order to construct pseudolinear functions, the next theorem can be useful.

THEOREM 1.2 (Rapcsák, 1991a). *If the functions $l(\mathbf{x}), \eta_i(f(\mathbf{x}))$, $i = 1, \ldots, n, \mathbf{x} \in A$, have continuous derivatives in all arguments on an open set $A \subseteq R^n$ and if* *they satisfy the compatibility conditions*

$$\begin{aligned}&\frac{\partial l(\mathbf{x})}{\partial x_j}\eta_i(f(\mathbf{x})) + l^2(\mathbf{x})\frac{d\eta_i(f(\mathbf{x}))}{df}\eta_j(f(\mathbf{x})) = \\ &\quad \frac{\partial l(\mathbf{x})}{\partial x_i}\eta_j(f(\mathbf{x})) + l^2(\mathbf{x})\frac{d\eta_j(f(\mathbf{x}))}{df}\eta_i(f(\mathbf{x})), \\ &\quad i, j = 1, \dots, n, \qquad \mathbf{x} \in A,\end{aligned} \tag{1.2}$$

then a uniquely determined continuously differentiable solution of the system

$$\frac{\partial f(\mathbf{x})}{\partial x_i} = l(\mathbf{x})\eta_i(f(\mathbf{x})), \qquad i = 1, \dots, n, \qquad \mathbf{x} \in A, \tag{1.3}$$

exists in a neighbourhood of every point of A when the value of the function f is prescribed at some point of the neighbourhood.

In the paper, the Fenchel problem of level sets is solved under the conditions that the boundaries of the nested family of convex sets in R^{n+1} are given by C^3 n-dimensional differentiable manifolds and the convex sets (so, the boundaries of the nested family of convex sets as well) determine an open or closed convex set in R^{n+1}.

2. Main results

The Fenchel problem of level sets consists of three parts to be solved in the smooth case:

(1) the common characterization of the equality level sets of a function and the boundaries of the nested family of convex sets as Riemannian n-dimensional submanifolds in R^{n+1},

(2) the characterization of the convex or generalized convex functions based on level sets and

(3) the construction of a convex or some generalized convex function by using the data of the nested family of convex sets so that the convex sets give the level sets.

Let R^{n+1} denote the $(n+1)$-dimensional Euclidean space which is a special Riemannian manifold endowed with the metric $G(\mathbf{x}) = I,\ \mathbf{x} \in R^{n+1}$, where I is the $(n+1) \times (n+1)$ identity matrix. Let us assume that a nested family of convex sets is given in R^{n+1} whose boundaries are C^3 (three times continuously differentiable co-ordinate functions) n-dimensional differentiable manifolds, and the convex sets determine an open convex set in R^{n+1}. First, the results are formulated by tensors, then by matrices based on the explicit formulation of tensors in co-ordinate representations.

Let M be an n-dimensional differentiable manifold and m a point in M. The tangent space TM_m at m is an n-dimensional vector space.

DEFINITION 2.1. A p-covariant tensor at m (for $p > 0$) is a real-valued p-linear function on $TM_m \times TM_m \times \ldots \times TM_m$ (p-copies). A tensor is symmetric if its value remains the same for all possible permutations of its arguments. A 0-covariant tensor at m is a real number. A tensor field on M is a mapping that assigns a tensor at m to each m in M.

DEFINITION 2.2. A second-order covariant tensor is positive semidefinite (definite) at a point $m \in M$ if the corresponding matrix is positive semidefinite (definite) on $TM_m \times TM_m$ in any co-ordinate representation. A second-order covariant tensor field is positive semidefinite (definite) on $A \subseteq M$ if it is positive semidefinite (definite) at every point of A.

In our case, the Riemannian metric is a positive definite symmetric second-order covariant tensor field. In R^{n+1}, the Euclidean metric induces a Riemannian metric on every submanifold so that the induced Riemannian metric be equal to the projection of the Euclidean one to every tangent space. In the case of an n-dimensional Riemannian submanifold in R^{n+1}, the normal space, orthogonal to the tangent space with respect to the Riemannian metric, is a one-dimensional subspace at every point, and the second fundamental tensor field, which is a second-order symmetric covariant tensor field (e.g., (Spivak, 1979)), provides the second-order approximation of the manifold related to the direction of the normal vector at every point.

THEOREM 2.1. *Let a three times continuously differentiable function f be defined on an open convex set $A \subseteq R^{n+1}$ such that $\nabla f(\mathbf{x}) \neq 0$, $\mathbf{x} \in A$, and assume that a nested family of convex sets in R^{n+1} is given so that the convex sets determine the open convex set $A \subseteq R^{n+1}$ to within a motion, and the boundaries of the convex sets are C^3 n-dimensional differentiable manifolds. Then, the equality level sets $M[f(\mathbf{x}_0)] = \{\mathbf{x} \in A |\ f(\mathbf{x}) = f(\mathbf{x}_0),\ \forall \mathbf{x}_0 \in A\}$ of a quasiconvex function f are equal to within a motion to the n-dimensional differentiable manifolds given by the boundaries of the nested family of convex sets in R^{n+1} iff*

(1) the induced Riemannian metrics of the equality level sets and the n-dimensional differentiable manifolds are identical at all the points of A,

(2) the second fundamental tensors of the equality level sets and the n-dimensional differentiable manifolds are identical and positive semidefinite at all the points of A and

(3) the Riemannian metrics and the second fundamental tensors satisfy the Gauss and the Codazzi-Mainardi equations (see formulas 3.2 and 3.3) at all the points of A.

The assumption, which ensures that the boundaries are differentiable manifolds, implies that the problem can be investigated in a co-ordinate neighbourhood of every point in the given n-dimensional differentiable manifolds. Thus, the boundaries of the nested family of convex sets in R^{n+1} can be studied in co-ordinate representations by the smooth mappings

$$\mathbf{x}(\mathbf{u};c) \in A \subseteq R^{n+1}, \quad \mathbf{u} \in U \subseteq R^n, \quad c \in J, \tag{2.1}$$

where U is an open set and J an interval. Let us assume that the range of f denoted by $rngf$ satisfies the condition $rngf = J$.

DEFINITION 2.3. In a given co-ordinate neighbourhood (2.1), the second fundamental quantities can be given in the form of

$$b_{ij}(\mathbf{u};c) = \left(\frac{\partial \mathbf{x}^2(\mathbf{u};c)}{\partial u_i \partial u_j}\right)^T \eta(\mathbf{u};c), \quad \mathbf{u} \in U, \quad c \in J, \quad i,j = 1,\dots,n, \tag{2.2}$$

where $\eta = (\eta_1,\dots,\eta_{n+1})$, $\eta_i : U \to R, \forall i$, are the normal vectors of the examined manifolds. The second fundamental forms are the quadratic forms defined by the matrix function of the second fundamental quantities on the tangent spaces of the given manifolds. The second fundamental forms are second-order symmetric covariant tensor fields on the manifolds.

THEOREM 2.1′. *Let a three times continuously differentiable function f be defined on an open convex set $A \subseteq R^{n+1}$ such that $\nabla f(\mathbf{x}) \neq 0$, $\mathbf{x} \in A$, and assume that a nested family of convex sets in R^{n+1} is given so that the convex sets determine the open convex set $A \subseteq R^{n+1}$ to within a motion, and the boundaries of the convex sets are C^3 n-dimensional differentiable manifolds. Then, the equality level sets $M[f(\mathbf{x})]$, $\forall \mathbf{x} \in A$, of a quasiconvex function f are equal to within a motion to the n-dimensional differentiable manifolds given by the boundaries of the nested family of convex sets in R^{n+1} iff in a convex co-ordinate neighbourhood (2.1) of every point of A, in the given differentiable manifolds*

(1) the induced Riemannian metrics of the equality level sets and the n-dimensional differentiable manifolds are equal to

$$G(\mathbf{x}(\mathbf{u};c)) = J_{\mathbf{u}}\mathbf{x}(\mathbf{u};c)^T J_{\mathbf{u}}\mathbf{x}(\mathbf{u};c), \qquad \mathbf{u} \in U \subseteq R^n, \quad c \in J, \tag{2.3}$$

(2) the matrices of the second fundamental tensors of the equality level sets and the n-dimensional differentiable manifolds are positive semidefinite and are equal to

$$B(\mathbf{x}(\mathbf{u};c)) = J_{\mathbf{u}}\mathbf{x}(\mathbf{u};c)^T H_{\mathbf{x}} f(\mathbf{x}(\mathbf{u};c)) J_{\mathbf{u}}\mathbf{x}(\mathbf{u};c),$$
$$\mathbf{u} \in U \subseteq R^n, \quad c \in J, \text{ and} \tag{2.4}$$

(3) the matrices of the Riemannian metric $G(\mathbf{x}(\mathbf{u};c))$, $\mathbf{u} \in U \subseteq R^n$, $c \in J$, and the matrices of the second fundamental tensors $B(\mathbf{x}(\mathbf{u};c))$, $\mathbf{u} \in U \subseteq R^n$, $c \in J$, satisfy the Gauss and the Codazzi-Mainardi equations

where $J_{\mathbf{u}}\mathbf{x}(\mathbf{u};c)$ is the Jacobian matrix of $\mathbf{x}(\mathbf{u};c)$ with respect to $\mathbf{u}$ at $\mathbf{u}$ and c, and $H_{\mathbf{x}} f$ is the Hessian matrix of the function f with respect to $\mathbf{x}$.

Since the open set $A \subseteq R^{n+1}$ provided by the given n-dimensional manifolds is an $(n+1)$-dimensional differentiable manifold, we can introduce tensor fields on A originated from the n-dimensional manifolds. This remark is important from the point of view of the geometric structure of the Fenchel problem of level sets, and allows to solve it under the conditions that the boundaries of the nested family of convex sets in R^{n+1} are C^3 n-dimensional differentiable manifolds and the boundaries of the nested family of convex sets determine an open convex set in R^{n+1}.

THEOREM 2.2. *Let a nested family of convex sets be given in R^{n+1} so that the boundaries of the convex sets are C^3 n-dimensional manifolds and the convex sets determine an open convex set A in R^{n+1}. If $\eta_i(\mathbf{x}) \in C^2(A, R), i = 1, \ldots, n+1, \eta = (\eta_1, \ldots, \eta_{n+1})$ is the twice continuously differentiable vector field on A originated from the normal vector fields of the given n-dimensional differentiable manifolds, there exists a function $l \in C^2(A, R)$ satisfying the compatibility conditions*

$$\frac{\partial l(\mathbf{x})}{\partial x_j}\eta_i(\mathbf{x}) + l(\mathbf{x})\frac{d\eta_i(\mathbf{x})}{\partial x_j} = \frac{\partial l(\mathbf{x})}{\partial x_i}\eta_j(\mathbf{x}) + l(\mathbf{x})\frac{d\eta_j(\mathbf{x})}{\partial x_i}, \qquad i, j = 1, \ldots, n+1, \qquad \mathbf{x} \in A \subseteq R^{n+1}, \tag{2.5}$$

of the system

$$\frac{\partial f(\mathbf{x})}{\partial x_i} = l(\mathbf{x})\eta_i(\mathbf{x}), \qquad i = 1, \ldots, n+1, \qquad \mathbf{x} \in A, \tag{2.6}$$

such that

$$B(\mathbf{x}) = (\nabla l^T(\mathbf{x})\eta(\mathbf{x}) + l(\mathbf{x})J\eta(\mathbf{x}))_{|TM}, \qquad \mathbf{x} \in A, \tag{2.7}$$

(the gradient of a function is a row vector and the symbol $|TM$ means the restriction to the tangent space of the given manifolds) is a positive semidefinite matrix function on A and the Gauss and the Codazzi-Mainardi equations hold, then a uniquely determined, twice continuously differentiable quasiconvex function f exists on A such that the equality level sets of f correspond to the boundaries of the nested family of convex sets when the value of the function f is prescribed at one point of A.

A quasiconvex function $f : A \to R$ with $\nabla f(\mathbf{x}) \neq 0$, $\forall \mathbf{x} \in A$, is pseudoconvex on an open convex set A, so if a condition ensures that a pseudoconvex function defined on an open convex set is convex transformable, then the Fenchel problem of level sets is solved.

DEFINITION 2.4. The augmented Hessian matrix of a twice continuously differentiable function $f : A \to R$ is defined by

$$H(f(\mathbf{x}); \rho(\mathbf{x})) = Hf(\mathbf{x}) + \rho(\mathbf{x})\nabla f(\mathbf{x})^T\nabla f(\mathbf{x}), \qquad \mathbf{x} \in A, \tag{2.8}$$

where $\rho : A \to R$ is a function.

A necessary condition for the convexifiability of function f over A is the existence of a function $\rho : A \to R$ such that $H(f(\mathbf{x}); \rho(\mathbf{x}))$ is a positive semidefinite matrix for all $\mathbf{x} \in A$. Sufficient conditions are given in Proposition 8.13 and Theorem 8.17 of Avriel, Diewert, Schaible and Zang (1988). We shall use one of them in the following statement.

COROLLARY 2.1. *If the conditions of Theorem 2.2 hold, and*

$$\mathit{rank}\,(\nabla l^T(\mathbf{x})\eta(\mathbf{x}) + l(\mathbf{x})J\eta(\mathbf{x})) \leq \mathit{rank}\, B(\mathbf{x}) + 1, \qquad \mathbf{x} \in A, \tag{2.9}$$

where the notation 'rank' means the rank of the corresponding matrix, then a twice continuously differentiable convex function f exists on A when the value of the function f is prescribed at one point of A.

COROLLARY 2.2. *If the nested family of convex sets determines a closed convex set A, the other conditions of Theorem 2.2, and condition (2.9) hold, then a convex function f exists on A when the value of the function f is prescribed at one point of A.*

Proof. In the case of a closed convex set A, the Fenchel problem of level sets can be solved in two steps. First, a smooth convex function can be constructed on the interior (or the relative interior) of the given closed convex set A by Theorem 2.2 and Corollary 2.1. Then, this convex function can be extended on the whole A to a convex function, by setting the values of the function f equal to $+\infty$ for all the boundary points of A. ■

We remark that another possibility for the extension of the convex function f defined on the interior of the set A is to close the epigraph of f by obtaining a lower semi-continuous function on A, then to assign the value $+\infty$ for the remaining boundary points.

3. Preliminary Lemmas and Theorems

In order to prove the theorems, we will use some co-ordinate representations of the given n-dimensional manifolds and the open convex set A which is an $(n+1)$-dimensional manifold.

DEFINITION 3.1. Let $A \subseteq R^n$ be an open convex set. A function $f : A \to R$ is said to be locally convex (pseudoconvex) if it is convex (pseudoconvex) in a convex neighbourhood of every point of A.

The following theorem is a Corollary of Theorem 2.2 in Rapcsák (1991), proved for the geodesic convex functions with respect to the Riemannian metrics.

THEOREM 3.1. *Let $A \subseteq R^n$ be an open convex set. Then, a function $f : A \to R$ is convex iff it is locally convex.*

Proof. If the Riemannian manifold $M = R^n$, and the Riemannian metric is equal to the Euclidean one in Theorem 2.2 [Rapcsák, 1991b], then the geodesic convex sets become convex sets, the geodesic convex functions convex functions and the property of Theorem 3.1 is proved. ■

The statement of Theorem 3.1 was proved for pseudoconvex functions by Komlósi (1983). Though the differential geometric theorems most important for us were expanded by Voss (1880), we will use the versions published in the book of Eisenhart (1964). The following statement ensures the existence of a non-null normal vector field for every given manifold. This result guarantees the consideration of nontrivial second fundamental forms in the direction of this vector field.

THEOREM 3.2 (Eisenhart, 1964, p.144). *The normals to an n-dimensional Riemannian manifold M with the fundamental tensor $G = (g_{ij})$ immersed in R^{n+1} form a null vector system iff the determinant of the matrices of G is zero in every co-ordinate representation.*

Now, the n-dimensional Riemannian submanifolds of R^{n+1} are characterized.

THEOREM 3.3. *(Voss, 1880; Eisenhart, 1964, p.187) In order that an n-dimensional Riemannian manifold M with the Riemannian metric $G = (g_{ij})$ be a real Riemannian submanifold of R^{n+1}, it is necessary and sufficient that the system of partial differential equations*

$$\sum_{\alpha=1}^{n+1} c_\alpha \frac{\partial x_\alpha(\mathbf{u})}{\partial u_i} \frac{\partial x_\alpha(\mathbf{u})}{\partial u_j} = g_{ij}(\mathbf{u}), \qquad i, j = 1, \ldots, n, \quad \mathbf{u} \in U \subseteq R^n, \quad (3.1)$$

admit a solution $\mathbf{x}(\mathbf{u}) \in R^{n+1}$, $\mathbf{u} \in U \subseteq R^n$, with a full rank Jacobian matrix for every co-ordinate neighbourhood of M.

The signs of the c's in (3.1) depend on the character of the tensor G. If the matrices of G are positive definite, then all the c's are equal to 1. The integrability conditions of the system of partial differential equations (3.1) are formulated in the following statement:

THEOREM 3.4 (Eisenhart, 1964, p.198). *In order that $G = (g_{ij})$ and $B = (b_{ij})$ be the first and second fundamental tensors of an n-dimensional Riemannian manifold M immersed in R^{n+1}, respectively, it is necessary and sufficient that the Gauss and the Codazzi-Mainardi equations satisfy, i.e.,*

$$R_{ijkl} = e(b_{ik}b_{jl} - b_{il}b_{jk}), \quad (3.2)$$

$$b_{ij,k} - b_{ik,j} = 0, \quad (3.3)$$

for $i, j, k, l = 1, \ldots, n$, where R_{ijkl} are the components of the Riemannian curvature tensor of M with respect to G, $b_{ij,k}$ means the covariant differentiation of the second fundamental quantities with respect to G and $e = 1$ or $e = -1$. Thus, the Riemannian manifold M is determined to within a motion in R^{n+1}.

LEMMA 3.1 (Rapcsák, 1994). *If f is a twice continuously differentiable function on an open set A and $\nabla f(\mathbf{x}) \neq 0$, $\mathbf{x} \in M[f(\mathbf{x}_0)]$, then*

$$Hf(\mathbf{x})_{|TM} = -B_{\nabla f/|\nabla f|}(\mathbf{x}), \quad x \in M[f(\mathbf{x}_0)], \tag{3.4}$$

where $B_{\nabla f/|\nabla f|}$ is the matrix of the second fundamental quantities of $M[f(\mathbf{x}_0)]$ in a co-ordinate representation and the index $|TM$ denotes restriction to the tangent space TM at the point $\mathbf{x}$.

The necessary part of the following theorem was first proved by Arrow and Enthoven (1961) and the sufficiency part by Crouzeix (1980).

THEOREM 3.5 (Arrow and Enthoven, 1961; Crouzeix, 1980). *Let f be a twice continuously differentiable function on the open convex set $A \subseteq R^n$ and suppose that $\nabla f(\mathbf{x}) \neq 0$ for every $\mathbf{x} \in A$. Then, f is quasiconvex on A iff*

$$\mathbf{x} \in A, \quad \mathbf{v} \in R^n, \quad \mathbf{v}^T \nabla f(\mathbf{x}) = 0 \quad \Rightarrow \quad \mathbf{v}^T Hf(\mathbf{x})\mathbf{v} \geq 0. \tag{3.5}$$

4. Proof of Theorem 2.1 and Theorem 2.1′

The gradient vector ∇f is different from zero on A, thus every equality level set is an n-dimensional differentiable manifold in R^{n+1}. R^{n+1} is a Riemannian manifold with $G(\mathbf{x}) = I$, $\mathbf{x} \in R^{n+1}$, which metric induces a Riemannian metric in every n-dimensional differentiable submanifold. A quasiconvex function $f : A \to R$ with $\nabla f(\mathbf{x}) \neq 0$, $\forall \mathbf{x} \in A$, is pseudoconvex on an open convex set A, so the problem can be considered in co-ordinate representations.

I. Necessary part. Assume that every equality level set coincides to within a motion with the corresponding n-dimensional differentiable manifold. If we introduce the induced Riemannian metric on the differentiable manifolds, then by Theorem 3.3, we obtain real n-dimensional Riemannian submanifolds in R^{n+1}, and that the induced Riemannian metrics are identical at all the points of A, furthermore, that they can be given by (3.1) in every co-ordinate representation. By Theorem 3.2, the normal vectors of every equality level set coincide with the normal vectors of the corresponding n-dimensional differentiable manifold, thus the second fundamental tensors are identical at all the points of A. By Lemma 3.1 and Theorem 3.5, the second fundamental tensors are positive semidefinite at all the points of A. By Theorem 3.4, the Riemannian metric and the second fundamental tensor satisfy the Gauss and the Codazzi-Mainardi equations at all the points of A.

II. Sufficiency part. Assume that conditions (1), (2) and (3) of Theorem 2.1 hold. By Theorem 3.4, this Riemannian metric and the second fundamental tensor field determine unambiguously an n-dimensional Riemannian submanifold to within a motion iff the Gauss and the Codazzi-Mainardi equations (3.2 and 3.3) hold at all the points of the given manifolds, so every equality level set coincides to within a motion with the corresponding n-dimensional manifold, and by Theorem 3.5, the function f is quasiconvex. ∎

5. Proof of Theorem 2.2

Due to the manifold structure of the boundaries of the nested family of convex sets, they can be studied in co-ordinate representations (2.1). A quasiconvex function $f : A \to R$ with $\nabla f(\mathbf{x}) \neq 0, \ \forall \mathbf{x} \in A$, is pseudoconvex on an open convex set A. By Theorem 3.1 and Komlósi's result (1983), both convex and pseudoconvex functions can be locally characterized, thus the Fenchel problem can be solved in co-ordinate representations.

By the Frobenius theorem (e.g., (Spivak, 1979)), a uniquely determined, twice continuously differentiable solution of system (2.6) exists in a neighbourhood of every point in A if compatibility conditions (2.5) hold on A and the value of function f is prescribed at some point of the neighbourhood. Now, it will be shown that it is enough to prescribe a value of f at one point only, in an arbitrary co-ordinate neighbourhood, then all the function values of f are determined on A. The proof is indirect. Let us assume that the value of f is prescribed at the point $\mathbf{x}_0 \in A$ and there exists a co-ordinate neighbourhood for which at least one value of f is not determined. Because the set A is convex, there exists a line segment between the point $\mathbf{x}_0 \in A$ and the point $\hat{\mathbf{x}} \in A$ where the value of f is not determined. The line segment is a compact set in A, thus there exists a finite number of co-ordinate neighbourhoods such that these cover the line segment. Moreover, a finite number of co-ordinate neighbourhoods can be chosen so that a chain of co-ordinate neighbourhoods be obtained and every member of this set should have a nonempty intersection with only one from the next, subsequent neighbourhoods in the direction of the vector $\hat{\mathbf{x}} - \mathbf{x}_0$. This is a contradiction, since all the values of the function f are determined on this line segment by this chain of co-ordinate neighbourhoods. It can be seen that two different chains of co-ordinate neighbourhoods with the same starting and end points determine the same function value at the end point.

By introducing the induced Riemannian metric on the boundaries of the nested family of convex sets, and by Theorem 3.3, we obtain n-dimensional Riemannian submanifolds - immersed in R^{n+1} - with the first fundamental tensor G, given by (2.3) in any co-ordinate representations (2.1). By Lemma 3.1, we have that the second fundamental tensor is determined by (2.7) in any co-ordinate representations. By Theorem 2.1, the equality level sets of the quasiconvex function f are equal to within a motion to the boundaries of the nested family of convex sets, from which the statement follows. ∎

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References

1. Arrow, K.J. and Enthoven, A.C., Quasi-concave programming, *Econometrica* **29** (1961) 779–800.
2. Avriel, M., Diewert, W. E., Schaible, S. and Zang, I., *Generalized Concavity*, Plenum Press, New York, London, 1988.
3. Crouzeix, J.P., Contributions a l'étude des fonctions quasiconvexes, Thèse, Université de Clermont-Ferrand, 1977.
4. Crouxeix, J.P., On second order conditions for quasiconvexity, *Mathematical Programming* **18** (1980) 349–352.
5. Debreu, G., *Representation of a Preference Ordering by a Numerical Function*, in: Decision Processes, Thrall, Coombs and Davis (eds.), John-Wiley and Sons, 1954.
6. Debreu, G., Least concave utility functions, *Journal of Mathematical Economics* **3** (1976) 121–129.
7. de Finetti, B., Sulle stratificazioni convesse, *Annali di Matematica Pura ed Applicata* **30** (1949) 173–183.
8. Eisenhart, L. P., *Riemannian Geometry*, Princeton University Press, Princeton, 1964.
9. Fenchel, W., *Convex Cones, Sets and Functions*, (mimeographed lecture notes), Princeton University Press, Princeton, New Jersey, 1953.
10. Fenchel, W., Über konvexe Funktionen mit vorgeschriebenen Niveaumannigfaltigkeiten, *Mathematische Zeitschrift* **63** (1956) 496–506.
11. Kannai, Y., Concavifiability and constructions of concave utility functions, *Journal of Mathematical Economics* **4** (1977) 1–56.
12. Kannai, Y., Concave utility functions – existence, constructions and cardinality, in: *Generalized concavity in optimization and economics*, (eds.): S. Schaible and Ziemba, W.T., Academic Press, New York (1981) 543–611.
13. Komlósi, S., Second order characterization of pseudoconvex and strictly pseudoconvex functions in terms of quasi-Hessians, in: *Contributions to the Theory of Optimization*, (ed.): F. Forgó, Department of Mathematics, Karl Marx University of Economics, Budapest (1983) 19–46.
14. Rapcsák, T., On pseudolinear functions, *European Journal of Operational Research* **50** (1991a) 353–360.
15. Rapcsák, T., Geodesic convexity in nonlinear optimization, *Journal of Optimization Theory and Applications* **69** (1991b) 169–183.
16. Rapcsák, T., On the connectedness of the solution set to nonlinear complementarity systems, *Journal of Optimization Theory and Applications* **81** (1994) 619–631.
17. Roberts, A. W. and Varberg, D. E., *Convex Functions*, Academic Press, New York, London, 1973.
18. Spivak, M., *A Comprehensive Introduction to Differential Geometry* I-V, Publish or Perish, Inc. Berkeley, 1979.
19. Voss, A., Zur Theorie der Transformation quadratischer Differentialausdrücke und der Krümmung höherer Mannigfaltigkeiten, *Mathematische Annalen* **16** (1880) 129–179.